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A decomposition theorem for neutrices

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ABSTRACT

Neutrices are convex additive subgroups of the nonstandard space $\mathbb{R}^k$, most of them are external sets. Because of the convexity and the invariance under some translations and multiplications, external neutrices are models for orders of magnitude. One dimensional neutrices have been applied to asymptotics, singular perturbations, and statistics. This paper shows that in $\mathbb{R}^k$, with standard k, every neutrix is the direct sum of k neutrices of $\mathbb{R}$. These components may be chosen to be orthogonal.

1. Setting and structure of this paper

The setting of this paper is the Internal Set Theory (*IST*) of Nelson [20,21]. *IST* is an axiomatic approach to Abraham Robinson's theory of infinitesimals which was originally based on a logical nonstandard model of the formal theory of analysis ("nonstandard analysis"). References [14,13] give up-to-date presentations and terminology in *IST*.

The formal language of this set theory contains two primitive symbols $\in$ and *st* (standard). Formulas that contain *st* are called external, and otherwise formulas are called internal and correspond to formulas of traditional set theory. The set of nonstandard real numbers $\mathbb{R}$ is defined in *IST* by the same formula that defines it in traditional set theory (a complete ordered field.) However, some $\alpha \in \mathbb{R}$ satisfy "$st\alpha$" (α is a "standard real number") and some do not.

Nelson's axioms restrict set formation to internal formulas, but we refer to a collection of internal elements satisfying a bounded external formula as an "external set". For example, the "set" of standard natural numbers is external, $\{x \in \mathbb{N} \mid stx\} \neq \mathbb{N}$. (A formula is bounded if quantifiers range over specific standard sets, like $\forall x \in \mathbb{N}(stx \to \ldots)$). Axiomatic approaches to external sets are given in [17].)

One important consequence of the *IST* axioms is the "Fehrele principle [23]": no galactic formula, i.e. a Σ_1-formula, starting with the "external quantifier" $\exists x(stx \wedge \ldots)$ is equivalent to a halic formula, i.e. a Π_1-formula, starting with the "external quantifier" $\forall x(stx \to \ldots)$. This principle allows us to prove the existence of a generalized supremum and constitutes a crucial step in the proof of the Orthogonal decomposition theorem for neutrices in two dimensions [26] (stated as Theorem 3.3).

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Classically, orders of magnitude are modeled with the O- or o-notation, which can be seen as additive groups of real functions in one variable [10]. External convex sets of nonstandard real numbers may be bounded without having an infimum and supremum, and may be invariant under at least some additions or translations. Simple examples are the external set (halo) of all infinitesimals, $\oslash = \{x \in \mathbb{R} \,||x| < 1/m$ for all standard $m \in \mathbb{N}\}$ and the external set (galaxy) of all limited real numbers, $\pounds = \{x \in \mathbb{R} \,||x| < m$ for some standard $m \in \mathbb{N}\}$. Such groups have been called neutrices in [18,19]. The term is borrowed from Van der Corput [27], who uses it to designate groups of functions, which may be more general than Oh's and oh's.

Definition 1.1. Let $k \in \mathbb{N}, k \geq 1$ be standard. A *neutrix* is a convex additive subgroup of the nonstandard space $\mathbb{R}^k$. When $k = 1$ we call the neutrix a *scalar neutrix*.

By convexity we mean to include nonstandard scalar multiples, so N is a neutrix if whenever $x, y \in N$ and $\lambda \in \mathbb{R}$ satisfies $0 < \lambda < 1$, then $-x \in N, x + y \in N$ and $\lambda x + (1 - \lambda)y \in N$. By external induction, if $x \in N$ and $m \in \mathbb{N}$ is a standard natural number, then $mx \in N$. If λ is a limited nonstandard real, i.e. $\lambda \in \pounds$, then $|\lambda| < m$ for some standard $m \in \mathbb{N}$, so by convexity, if $x \in \mathbb{N}$ and λ is a limited scalar, $\lambda x \in \mathbb{N}$. In other words, a subset of $\mathbb{R}^k$ is a neutrix if and only if it is an $\pounds$ -submodule of $\mathbb{R}^k$.

The only internal neutrices are internal linear subspaces of $\mathbb{R}^k$. If α is a nonstandard real and N is a neutrix, then αN is also a neutrix. The external set of infinitesimals, $\oslash$ and the external set of limited nonstandard real numbers, £, are scalar neutrices. If ϵ is a fixed positive infinitesimal, the set $\epsilon\pounds \subset \oslash$ because $\epsilon\alpha = \sqrt{\epsilon}$ makes $\alpha = 1/\sqrt{\epsilon}$, unlimited. Similarly, $\frac{1}{\epsilon}\oslash \supset \pounds$ because $\sqrt{\epsilon} \simeq 0$, but $\frac{\sqrt{\epsilon}}{\epsilon} = 1/\sqrt{\epsilon}$ is unlimited. (We use $\subset$ for strict inclusion and $\subseteq$ for included or equal.)

The external set of unlimited numbers is denoted $\not\infty$. The external set of positive "appreciable numbers" or positive non-infinitesimal, limited numbers, is denoted $@ = \left\{\lambda \in \pounds \,\middle|\, \lambda \gtrapprox 0\right\}$. Some other scalar neutrices can be defined in terms of these non-neutrix "order of magnitude" sets. Fix a positive infinitesimal ϵ. The set we denote $\pounds\epsilon^{\not\infty}$ of all numbers smaller in magnitude than every standard power of ϵ is a neutrix. (Note that if $|x| < \lambda\epsilon^m$ for every standard m then by axiom I , there is an unlimited n so that $|x| < \lambda\epsilon^n$.) The neutrix we denote $\pounds e^{-@/\epsilon}$ contains all numbers satisfying $|x| < \lambda e^{-a/\epsilon}$ for some limited λ and appreciable a. These two neutrices are not scalar multiples of either $\oslash$ or £. (See [23].)

External numbers [18,19] are the sum of a (nonstandard) real number and a scalar neutrix. Neutrices and external numbers have occurred in various settings (the terminology not being explicit in all papers): singular perturbation theory (in particular as thicknesses of boundary layers [1,11,12,2–4]) asymptotic approximations (domains of validity of asymptotic behavior and reasoning [23]), probability theory (mass and queue of probability distributions [24]) and mathematical psychology (imperfect knowledge of maximal utility [25]). Special mention has to be made of the work of Bosgiraud, who applies the external numbers to problems of modeling and calculation of insecure statistical events in a series of papers [5–9]. The external numbers satisfy a calculus, which is not unlike the calculus of the reals. We refer to [18] or [19] for definitions.

The main result of this article is as follows.

Theorem 1.2 (*Orthogonal Decomposition Theorem*). *Let $k \in \mathbb{N}, k \geq 1$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix. Then there are neutrices $N_1 \supseteq \cdots \supseteq N_k$ in $\mathbb{R}$ and orthonormal vectors $u_1, \ldots, u_k$ such that*

$$N = N_1u_1 \oplus \cdots \oplus N_ku_k.$$

Moreover, if $M_1 \supseteq \cdots \supseteq M_k$ in $\mathbb{R}$ are neutrices and $v_1, \ldots, v_k$ are orthonormal vectors with $N = M_1v_1 \oplus \cdots \oplus M_kv_k$, it holds that

$$M_1 = N_1, \ldots, M_k = N_k.$$

The Orthogonal decomposition theorem is false in the case where k is unlimited, as we show in Section 6.

On the basis of the Orthogonal decomposition theorem it could be interesting to develop fragments of linear algebra, matrix-calculus, geometry and multivariate analysis and statistics, in order to model approximate phenomena or orders of magnitude in several dimensions. An example in the complex domain where $\left(1 + \frac{z}{n}\right)^n$ approximates e^z is given in the last section and hints at how this might develop.

This article extends the paper [26], where the result is proved for $k = 2$, its proof being based on a sort of generalized Dedekind completeness argument (see below). The proof of the general case bears almost no resemblance to it and is based on external induction and orthogonal projections. The Section 3 outlines the two dimensional result. Section 5 contains the proof of the Orthogonal decomposition theorem in standard dimension, with some preliminary properties on orthogonality and near-orthogonality developed in Section 4. We start by considering some algebraic aspects of the Orthogonal decomposition theorem, in the following section.

2. Fundamental and algebraic aspects of the Orthogonal decomposition theorem

We emphasize that the Orthogonal decomposition theorem is meant to be a particular theorem of nonstandard linear algebra and analysis, and is seen as a possible tool for approximate ordinary more-variable calculus on problems which result form mathematical models of uncertainties. This aim motivates a decomposition in orthogonal directions, which is perhaps the most relevant for such rather plain mathematics.

The algebraic setting which comes the most close to the Orthogonal decomposition theorem is the theory of modules over a given non-noetherian ring R [28,15]. Two notions – with their respective variants and weakenings - appear to be particularly relevant for finite decomposition theorems: maximality of the ring R for the existence of decompositions and locality of the ring of endomorphisms of the indecomposable components.

Below we show that:

1. £ is S-maximal. S-maximality is an adaptation of maximality to the external setting.
2. The ring of endomorphisms of a scalar neutrix is local.

These two algebraic properties appear to be essentially equivalent to the Generalized completeness theorem for external cuts, which was the principal tool used in the proof of the existence of the orthogonal decomposition in two dimensions in [26]. We repeat here this completeness theorem, which extends the usual completeness of $\mathbb{R}$ to external cuts.

Theorem 2.1 (*Generalized Completeness Theorem [23]*). *Let (A, B) be a (possibly) external cut. Then there exists $x \in \mathbb{R}$ and a unique scalar neutrix K such that either $A = (-\infty, x + K]$ or $A = (-\infty, x + K)$.*

Its proof uses the full saturation of *IST*. The two algebraic properties mentioned above open the possibility to adapt classical decomposition theorems to neutrices. However:

3. There are some foundational obstacles for adaptation of classical decomposition theorems and their proofs.
4. The Orthogonal decomposition theorem is finer, both with respect to existence and uniqueness.

The discussion below will require more formalization of external sets than the actual proof of the Orthogonal decomposition theorem. We follow the approach by Kanovei and Reeken in [17]. They consider the system *BST*, which are all formulas of *IST* bounded by standard sets. This restriction does not affect the amount of saturation needed to prove the Generalized completeness theorem. Concrete external sets with internal elements may be seen as abbreviations of formulas, having, due to Nelson's Reduction Algorithm, the particular form $\forall^{st} y \in Y \exists^{st} z \in Z \phi(x, y, z)$, where ϕ is an internal formula and Y and Z are standard sets. "Naive" operations, like union and intersection are formalized in the system *EEST*. External sets with external elements obey the system *HST*, and adaptation of Hrbáček's system [16] to *BST*. The axioms for external sets of the system *HST* are essentially weaker than – mutatis mutandis – *ZFC*, due to the so-called Hrbáček's paradox: the requirement of full saturation is not compatible with the power-set axiom and choice. Absence of the latter axioms have some negative consequences for forming quotient groups and sets of representatives of the classes of equivalence.

Also, some algebraic properties refer to finiteness. In the context of external sets they must be reformulated in terms of S-finiteness, i.e. the cardinalities in question should only be standard natural numbers.

It is supposed that the algebraic notions in consideration respect the above mentioned restrictions, and to distinguish them from the classical algebraic notions, if necessary, we state them as S-notions.

To start with, we show that £ is not S-noetherian. Indeed, let $\omega \in \mathbb{R}$ be positive unlimited, and put $G_m = \omega^m £$. Then $G_m \nsubseteq G_{m+1}$ for all indices m. Consider ω^ω which is not an element of G_m for all standard indices m. Let $\epsilon \in £$ be sufficiently small such that also $\epsilon\omega^\omega \in £$ (for example $\epsilon = 1/\omega^{\omega^\omega}$). Then $(\epsilon G_m)_{stm}$ is a strictly ascending chain of £-submodules of N.

The union $G \equiv \cup_{stm} G_m$ is a scalar neutrix not isomorphic, like all of the G_m, to £ [23]. Such constructions by strictly ascending or descending chains (including of higher cardinality) imply that there is a proliferation of scalar neutrices which are nonisomorphic. The richness in substructures hints that, generally spoken, there is no "easy" way to derive decompositions of neutrices $N \subseteq \mathbb{R}^k$ for $k \geq 2$ standard. It is not difficult to verify that N does not have an S-finite dimension in the sense of vectorspaces over ${}^\sigma\mathbb{R}$, and that the only non-zero neutrices in $\mathbb{R}^k$, which are S-finitely generated, are of the form $N = £v_1 \oplus \cdots \oplus £v_p$ with $v_1, \ldots, v_p$ orthogonal – possibly nonstandard both in norm and direction - vectors and $1 \leq p \leq k$.

2.1. The external set of limited numbers as S-maximal valuation domain

Definition 2.2. A *valuation domain* is an integral domain such that the set of its ideals is a chain under inclusion.

We verify that £ is an S-valuation domain. First, it has no zero-divisors other than 0. Second, we observe that any S-ideal (defined in *EEST*) is a scalar neutrix, and vice-versa. Further, scalar neutrices M and N are ordered by inclusion, because there can not be elements $m \in M\backslash N$ and $n \in N\backslash M$. If there were, and $|m| = \min(|m|, |n|)$, then $[-m, m] \subseteq M \cap N$, since by convexity, $[-m, m] \subseteq [-n, n] \subseteq M \cap N$, but then $m \in N$. The contradiction shows that $M \cap N$ equals either M or N. We may write $M \cap N = \min(M, N)$ to indicate this.

Definition 2.3. Let R be a *valuation domain*. It is called *maximal* if every family of sets $\{a_i + L_i \,|i \in I\}$, where $a_i \in R$ and L_i is an ideal of R for all elements i of some index set I, which satisfies the finite intersection property, has non-empty intersection.

We show that the Generalized completeness theorem implies the S-maximality of £ and vice-versa.

Observe that if $a \in £$ and L is an ideal of £ the external set $a + L$ is an external number [19]. It is algebraically obvious that the non-empty intersection of two external numbers is an external number. By external induction the non-empty intersection of an S-finite set of external numbers is equal to one of these external numbers.

Let I be a possibly external index set and $F \equiv \{\alpha_i \subseteq £ \,|i \in I\}$ be a family of external numbers with the S-finite intersection property. Clearly $\cap F \neq \emptyset$ if $\cap F = \alpha_j$ for some $j \in I$. In the remaining case for all $i \in I$ there exists $k \in I, k \neq i$ such that

$\alpha_k \subset \alpha_i$. As a consequence $\cap F \subset a + \oslash$ for some limited a. So, up to addition with $-a$, we need only to prove the property of non-empty intersection for families F such that all $\alpha_i \subset \oslash$. We will call such families *infinitesimal*.

Let $F \equiv \{\alpha_i \subset \oslash \,|\, i \in I\}$ be an infinitesimal family of external numbers. For all $i \in I$ we have $\alpha_i = a_i + N_i$, where $a_i \simeq 0$ and N_i is a scalar neutrix. We define

$$N = \bigcap_{i \in I} N_i$$
$$U = \{y \simeq 0 \,|\, \exists i \in I, y > \alpha_i\}$$
$$D = \{y \simeq 0 \,|\, \exists i \in I, y < \alpha_i\}.$$

Observe that none of the external sets N, U or D is empty.

Theorem 2.4. *Let $F \equiv \{\alpha_i \subset \oslash \,|\, i \in I\}$ be an infinitesimal family of external numbers.*

1. $D < U$.
2. $(\forall \epsilon > N)(\exists y \in D)(\exists z \epsilon U)(z - y < \epsilon)$.
3. $N = \{\delta \in \mathbb{R} \,|\, D + \delta = D\} = \{x \in \mathbb{R} \,|\, U + \delta = U\}$.
4. $(\exists x \simeq 0)(D = [\oslash, x + N) \wedge U = (x + N, \oslash]$.
5. $(\exists x \simeq 0)(\cap F = x + N)$.
6. *£ is an S-maximal valuation domain.*

Proof. 1. Let $y \in D$ and $z \in U$. Let $i \in I$ be such that $y < \alpha_i$ and $j \in I$ be such that $\alpha_j < z$. If $\alpha_i \cap \alpha_j = \alpha_i$ we have $y < \alpha_i < z$ and if $\alpha_i \cap \alpha_j = \alpha_j$ we have $y < \alpha_j < z$. Hence $y < z$.
2. Let $\epsilon > N$, say $\epsilon > N_i$, where $i \in I$. Let $a_i \in \alpha_i$. Put $y = a_i - \epsilon/2$ and $z = a_i + \epsilon/2$. Then $y < \alpha_i < z$. Hence $y \in D, z \in U$ and $z - y = \epsilon$.
3. Let $y \in D$. Let $i \in I$ be such that $y < \alpha_i$. Then $y + N_i < \alpha_i$. So $y + N < \alpha_i$. Hence $D + N \subseteq D$. As a consequence $N \subseteq \{\delta \in \mathbb{R} \,|\, D + \delta = D\}$. It follows from part 2 that $\{\delta \in \mathbb{R} \,|\, D + \delta = D\} \not\supset N$. We conclude that $N = \{\delta \in \mathbb{R} \,|\, D + \delta = D\}$. The proof that $N = \{x \in \mathbb{R} \,|\, U + \delta = U\}$ is analogous.
4. It follows from part 3 and the Generalized completeness theorem that there exists $x \simeq 0$ such that $D = [\oslash, x + N)$ or $D = [\oslash, x + N]$. In the latter case $x \in D$, which means that for some $i \in I$, one has $x < \alpha_i$. But then $x + N_i < \alpha_i$, which implies that $x + N_i \subseteq D$. Since $N_i \supset N$, we derive that $D \cap U \neq \emptyset$, a contradiction with Part 1. Hence $D = [\oslash, x + N)$. In analogous way one proves that $U = (\xi + N, \oslash]$ for some $\xi \simeq 0$. Suppose $\xi - x \notin N$. Then $\epsilon \equiv \xi - x > N$ by Part 1. Let $y \in D$ be such that $y + \epsilon \in U$. Then $y + \epsilon > \xi + N$, so $y > x + N$, a contradiction. Hence $\xi - x \in N$. This concludes the proof of part 4.
5. From part 4.
6. From part 5. ■

The proof of the two-dimensional case of the Orthogonal decomposition theorem in [26] considered analogous external sets $D, U \subset \oslash$, leaving out a "hole" in the form of an external number. In Section 3 we give a new, simple proof of the two-dimensional case on the basis of the above theorem.

We now show the converse to Theorem 2.4. We write $\mathbb{E}$ the external set of all external numbers. For $\alpha \in \mathbb{E}$ we write N_α the neutrix-part of α, i.e. $N_\alpha = \{\lambda \in \mathbb{R} \,|\, \alpha + \lambda = \alpha\}$.

Theorem 2.5. *The S-maximality of £ implies the Generalized completeness theorem.*

Proof. Let (G, H) be a cut of $\mathbb{R}$. Let $K = \{\delta \in \mathbb{R} \,|\, G + \delta = G \wedge H + \delta = H\}$. Then K is clearly a scalar neutrix. Up to a change of scale we may suppose that $K \subset \oslash$. Define $\mathcal{F} \subseteq \mathbb{E}$ by

$$\mathcal{F} = \{\alpha \in \mathbb{E} \,|\, \alpha \cap G \neq \emptyset \wedge \alpha \cap H \neq \emptyset\}.$$

Let $n \in \mathbb{N}$ be standard and $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$. One shows by external induction that $\alpha_1, \ldots, \alpha_n$ have a common interval, i.e. $\cap_{1 \le i \le n} \alpha_i \neq \emptyset$. Then $\beta \equiv \cap \mathcal{F} \neq \emptyset$ by the S-maximality of £. Let $x \in \cap \mathcal{F}$ and $N = \cap_{\alpha \in \mathcal{F}} N_\alpha$ Then N is a scalar neutrix and

$$\beta = \cap \{\alpha \,|\, \alpha \in \mathcal{F}\} = \cap \{x + N_\alpha \,|\, \alpha \in \mathcal{F}\} = x + \cap \{N_\alpha \,|\, \alpha \in \mathcal{F}\} = x + N.$$

Hence $\beta \in \mathbb{E}$. Observe that $y < \beta$ implies that $y \in G$ and that $z > \beta$ implies that $z \in H$. We distinguish the cases $\beta \notin \mathcal{F}$ and $\beta \in \mathcal{F}$.

1. $\beta \notin \mathcal{F}$: Then $\beta \subseteq G$ or $\beta \subseteq H$. In the first case $G = (-\infty, \beta]$ and in the second case $H = [\beta, \infty)$, hence $G = (-\infty, \beta)$. We observe that the case $\beta \in \mathbb{R}$ (i.e. $N = \{0\}$) corresponds to the usual Dedekind completeness of $\mathbb{R}$.
2. $\beta \in \mathcal{F}$: We prove that there exists $\epsilon > 0$ such that $N = \epsilon £$ and $K = \epsilon \oslash$. If not, there exists a scalar neutrix M such that $K \subset M \subset N$. Because $K \subset M$ there are $\xi \in G$, $\eta \in H$ with $\eta - \xi \in M$. Hence $\xi + M \in \mathcal{F}$ and $\xi + M \subset \xi + N = \beta$, a contradiction. Define $G_{x,\epsilon} = (-x + G)/\epsilon$ and $H_{x,\epsilon} = (-x + H)/\epsilon$. Because $(G_{x,\epsilon}, G_{x,\epsilon})$ is a cut of $\mathbb{R}$, by the Axioms of Standardization and Transfer $({}^sG_{x,\epsilon}, {}^sH_{x,\epsilon})$ is a standard cut of $\mathbb{R}$. Now $G_{x,\epsilon} \cap £ \neq \emptyset$ and $H_{x,\epsilon} \cap £ \neq \emptyset$, so ${}^sG_{x,\epsilon}$ and ${}^sH_{x,\epsilon}$ are non-empty. Let $s = \sup {}^sG_{x,\epsilon}$. Then s is standard. Also $s \in {}^sG_{x,\epsilon} \Leftrightarrow {}^sG_{x,\epsilon} = (-\infty, s] \Leftrightarrow G_{x,\epsilon} = (-\infty, s + \oslash]$ and $s \in {}^sH_{x,\epsilon} \Leftrightarrow {}^sH_{x,\epsilon} = [s, \infty) \Leftrightarrow H_{x,\epsilon} = [s + \oslash, \infty)$. In the first case $G = (-\infty, x + \epsilon s + \epsilon\oslash]$ and in the second case $H = [x + \epsilon s + \epsilon\oslash, \infty)$, hence $G = (-\infty, x + \epsilon s + \epsilon\oslash)$. ■

The equivalence of generalized completeness and the non-empty intersection property for families of external numbers satisfying the finite-intersection property may be compared to a similar classical equivalence: one uses Dedekind-completeness to prove that the intersection of a nested sequence of intervals of real numbers is non-empty, and again the nonempty intersection of a nested sequence of intervals of real numbers may be used to prove Dedekind-completeness.

2.2. S-locality of the external ring of endomorphisms of scalar neutrices

Definition 2.6. A ring is *local* if it has it has a unique maximal ideal.

Clearly £ is local, with maximal ideal $\oslash$. Let N be a scalar neutrix. We verify that the ring E of all external £ -module endomorphisms of N is local. Let $\phi : N \to N$ be such an endomorphism. We show that there exists $\lambda \in \mathbb{R}$ such that $\phi(x) = \lambda x$ for all $x \epsilon N$; this means in particular that ϕ is the restriction to N of an internal homomorphism. The property is immediate if $N = \{0\}$, so we may assume that N contains some non-zero element n. Let $\phi : N \to N$ be an (external) endomorphism. Let $\lambda = \phi(n)/n$. Let $x \in N, x \neq 0$. If $|x| \leq |n|$, we have $\phi(x)/x = \phi(nx/n)/x = x/n \cdot \phi(n)/x = \phi(n)/n$ and if $|x| > |n|$, we have $\phi(n)/n = \phi(nx/x)/n = n/x \cdot \phi(x)/n = \phi(x)/x$. Then $\phi(x) = \lambda x$ for all $x \in N$. Hence we may identify E with $N : N = \{\lambda \in \mathbb{R} \,|\lambda N \subseteq N\}$. This scalar neutrix is a ring with unity and unique maximal ideal $M \equiv \{\lambda \,|\lambda N \subset N\}$; both $N : N$ and M were determined by set-theoretic and by algebraic means in terms of logarithmic and exponential functions applied to N [19,26]. An essential tool was that every external scalar neutrix is isomorphic to a (unique) idempotent neutrix I, i.e. $N = \lambda I$, where $I \cdot I = I$. As a consequence the ring of endomorphisms of N is S-local.

The latter isomorphism property is a consequence of the Generalized completeness theorem. As far as external sets are concerned, the Generalized completeness theorem also follows from the isomorphism property.

Theorem 2.7. *Assume every external scalar neutrix is isomorphic to a unique idempotent neutrix I. Then the Generalized completeness theorem holds for external cuts.*

Proof. Let (A, B) be an external cut. Let $K = \{x \in \mathbb{R} \mid x + A = A \wedge x + B = B\}$. Then K is an external scalar neutrix. Let $\omega > 0$ be such that $\omega K \supseteq$ £. Let $\widehat{A} = \omega A$ and $\widehat{B} = \omega B$. Let $N = [-\exp \widehat{A}, \exp \widehat{A}]$. Then

$$£N = [-\exp \widehat{A} + £^-, \exp \widehat{A} + £^+] = [-\exp \widehat{A}, \exp \widehat{A}].$$

So N is a scalar neutrix. Let I be an idempotent neutrix and $\lambda > 0$ be such that $N = \lambda I$. We distinguish the cases (1) $1 \in I$ and (2) $1 \notin I$.

1. $1 \in I$: One has $\lambda \in N$, so $\log \lambda \in \widehat{A}$. Let $\widetilde{I} = \{x \in I \,|x \geq 1\}$. Because I is idempotent, it holds that $\widetilde{I}^{£^+} = \widetilde{I}$. So $£^+ \log \widetilde{I} = \log \widetilde{I}$. Hence $G \equiv -\log \widetilde{I} \cup \log \widetilde{I}$ is a scalar neutrix. We have $\widehat{A} = (-\infty, \log \lambda + G]$, hence $A = (-\infty, \frac{\log \lambda}{\omega} + \frac{G}{\omega}]$.
2. $1 \notin I$: One has $\lambda > N$, so $\log \lambda \in \widehat{B}$. Let $J = \{x \in \mathbb{R} \,|N < \lambda x \leq 1\}$. It is not difficult to see that the idempotent neutrix I is a prime ideal of £. Hence $J \cdot J = J$, which implies that $J^{£^+} = J$. So $£^+ \log J = \log J$. Hence $H \equiv \log J \cup -\log J$ is a scalar neutrix. We have $\widehat{B} = [\log \lambda + H, \infty]$, hence $A = (-\infty, \frac{\log \lambda}{\omega} + \frac{H}{\omega})$.

   The uniqueness of G and H follows from the uniqueness of I. ∎

2.3. On classical decomposition theorems

The Orthogonal decomposition theorem of neutrices, if seen as modules over £, comes perhaps most close to classical decomposition theorems in terms of so-called uniserial modules.

Definition 2.8. A module M is called *uniserial* if the set of all its submodules is totally ordered by inclusion.

Clearly all scalar neutrices are uniserial £-modules. If $k \geq 2$, submodules N which are uniserial are of the form $N = Mv$, where v is a non-zero vector and M is a scalar neutrix. All £-modules N of $\mathbb{R}^k$ which are not of this form are not uniserial. Indeed, the Orthogonal decomposition theorem yields at least two scalar non-zero neutrices N_1, N_2, with corresponding orthonormal vectors u_1, u_2 , such that $N_1u_1, N_2u_2 \subset N$, while neither $N_1u_1 \subseteq N_2u_2$ nor $N_1u_1 \subseteq N_2u_2$.

Uniserial modules are indecomposable in the sense that they cannot be a direct sum of two submodules: one would be included into the other, making any sum equal to the larger one, i.e. not a direct sum. So in the absence of finite generation, uniserial modules, in a sense, play the same role as vectorspaces of dimension one.

This suggests to consider the following algebraic decomposition theorem:

Theorem 2.9 (*Algebraic Decomposition Theorem*). *For $k \in \mathbb{N}$ standard, an £-module $N \subseteq \mathbb{R}^k$ is the direct sum $\mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_k$ of k uniserial £-modules. The uniserial £-modules are unique up to isomorphism.*

Below we present some algebraic settings for this type of decomposition theorem and discuss the relation with the Orthogonal decomposition theorem.

Definition 2.10. Let R be a ring and M be a module over R.

1. The module M is said to be *monoserial* if it can be embedded in a finite direct sum of uniserial modules.
2. The module M is called *weakly polyserial* if there exists a sequence of submodules $\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that for all i with $1 \leq i \leq n$ the quotients M_i/M_{i-1} are uniserial.

Let $k \in \mathbb{N}$ be standard. Any neutrix $N \subseteq \mathbb{R}^k$ is S-monoserial, because it is an external subset of $\mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_k$, where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$. So Theorems 1.2 and 2.9 are special cases of a decomposition of a monoserial module into a direct sum of uniserial modules.

The more general notion of polyseriality is perhaps the most relevant for the Algebraic decomposition theorem, were it not for a serious limitation due to weakness of axiomatics for external sets. We follow [15]. Theorem XII 2.2 implies that submodules of direct sums of uniserial modules are weakly polyserial (this means that monoserial modules are weakly polyserial) and Proposition XII 2.4 states that a weakly polyserial module over a maximal valuation domain is a direct sum of uniserial modules. Note that a neutrix is an S-monoserial module, hence it should be S-weakly polyserial, and that £ is a S-valuation domain. Still, it is not obvious how to define S-weakly polyserial properly. Indeed, the definition of quotient structures needs the power-set axiom, which appears to be not totally compatible with full saturation. The problem may be illustrated by the following. Consider the equivalence relation $xSy \Leftrightarrow x - y \simeq 0$. When applied to £, the quotient set may be identified with $\{x + \oslash \,|x \in {}^{\sigma}\mathbb{R}\}$. When applied to $\mathbb{R}$, such an identification is still possible with (say) $\{x + \oslash\,|x \in \mathbb{N}+{}^{\sigma}\mathbb{R}\}$. Such constructions seem less evident for other equivalence relations, like $xLy \Leftrightarrow x - y$ is limited.

Let us illustrate the effects of the Orthogonal decomposition theorem and the results on decomposition of polyserial modules mentioned in [15] with the following example. Let $\omega \simeq \infty$. Consider

$$N = £\begin{pmatrix}\omega\\1\end{pmatrix} \oplus \oslash\begin{pmatrix}0\\1\end{pmatrix}.$$

Note that N is a sort of diagonal in $\omega£\binom{\omega}{0} \oplus £\binom{0}{1}$. The Orthogonal decomposition theorem yields the orthonormal decomposition

$$N = \omega£\begin{pmatrix}\frac{\omega}{\sqrt{\omega^2+1}}\\ \frac{1}{\sqrt{\omega^2+1}}\end{pmatrix} \oplus \oslash\begin{pmatrix}\frac{-1}{\sqrt{\omega^2+1}}\\ \frac{\omega}{\sqrt{\omega^2+1}}\end{pmatrix}.$$

The method of theorem XII 2.2 starts with an arbitrary decomposition of the module of reference $\mathbb{R}^2$, say $\mathbb{R}\binom{1}{0} \oplus \mathbb{R}\binom{0}{1}$. The decomposition of N is made on the basis of $N\cap\mathbb{R}\binom{1}{0} / \{0\} = N\cap\mathbb{R}\binom{1}{0} = \oslash\omega\binom{1}{0}$, and $N\cap\mathbb{R}\binom{1}{0}\oplus\mathbb{R}\binom{0}{1} \Big/ N\cap\mathbb{R}\binom{1}{0} = N \Big/ \oslash\omega\binom{1}{0}$, which, if allowed, we may identify with £ or $£\binom{0}{1}$. We observe that the scalar neutrices $\omega£$ and $\oslash$ are determined up to isomorphism indeed, but that the corresponding directions, more precisely – the internal subspaces $\mathbb{R}\binom{\omega}{1}$ and $\mathbb{R}\binom{-1/\omega}{1}$ – are by no means specified, while especially the first one is essential for understanding the shape of the neutrix.

Still there is a relation between the two decompositions 1.2 and 2.9. The decomposition into uniserial submodules implies the existence of a finer orthogonal decomposition, and the uniqueness of the orthogonal components implies uniqueness up to isomorphism of the uniserial submodules. We start with two lemmas.

Lemma 2.11. *Let $k \in \mathbb{N}$ be standard. Let $N \subseteq \mathbb{R}^k$ be a neutrix. Assume $N = M_1v_1 \oplus \cdots \oplus M_kv_k$, where $M_1, \ldots, M_k$ are scalar neutrices and $v_1, \ldots, v_k$ are linearly independent vectors. Let i with $1 \le i \le k$ and p be the orthogonal projection on $\mathbb{R}{v_i}^{\perp}$. Then $p(N) = \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j)$.*

Proof. Because p is linear and $p(M_iv_i) = 0$, one has $p(N) = \sum_{1\le j\le k, j\ne i} M_jp(v_j)$. It follows from ordinary linear algebra that $(p(v_2), \ldots, p(v_k))$ is linearly independent. Hence $p(N) = \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j)$. ■

Lemma 2.12. *Let $k \in \mathbb{N}$ be standard. Let $N \subseteq \mathbb{R}^k$ be a neutrix. Assume $N = M_1v_1 \oplus \cdots \oplus M_kv_k$, where $M_1, \ldots, M_k$ are scalar neutrices and $v_1, \ldots, v_k$ are linearly independent vectors. Let i with $1 \le i \le k$ be such that $\|M_iv_i\| = |L|$. Let p be the orthogonal projection on $\mathbb{R}{v_i}^{\perp}$. Then $N = M_iv_i \oplus \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j)$.*

Proof. Let p_i be an orthogonal projection on $\mathbb{R}v_i$. Let $n \in N$. Then $\|p_i(n)\| \le \|n\| \in L$, so $p_i(n) \in M_iv_i$. Because N is a neutrix $p(n) = n - p_i(n) \in N$. Hence $p(N) = \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j) \subseteq N$. Again because N is a neutrix $M_iv_i \oplus \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j) \subseteq N$. Because $n = p(n) + p_i(n)$ also $N \subseteq M_iv_i \oplus \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j)$. Hence $N = M_iv_i \oplus \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j)$. ■

Theorem 2.13. *Let $k \in \mathbb{N}$ be standard. Let $N \subseteq \mathbb{R}^k$ be a neutrix. Assume $N = M_1v_1 \oplus \cdots \oplus M_kv_k$, where $M_1, \ldots, M_k$ are scalar neutrices and $v_1, \ldots, v_k$ are linearly independent vectors. Then there exist scalar neutrices $N_1 \cong M_1, \ldots, N_k \cong M_k$ and orthonormal vectors $u_1, \ldots, u_k$ such that $N = N_1u_1 \oplus \cdots \oplus N_ku_k$.*

Proof. By external induction in k. If $k = 1$, take $u_1 = \frac{v_1}{\|v_1\|}$ and $N_1 = M_1\|v_1\|$. Then $M_1v_1 = N_1u_1$ and $N_1 \cong M_1$. Assume the theorem is proved for $k - 1$, where $k \ge 2$ is standard. We prove the theorem for k. Let i with $1 \le i \le k$ be such that $\|M_iv_i\| = |L|$. Let u_i be a unit vector and $N_i \cong M_i$ be a scalar neutrix such that $M_iv_i = N_iu_i$. Let p be the orthogonal projection on $\mathbb{R}{u_i}^{\perp}$. Then $N = N_iu_i \oplus \bigoplus_{1\le j\le k, j\ne i} M_jp(v_j)$ by Lemma 2.12. By the induction hypothesis there are non-zero neutrices $N_j \cong M_j$ and orthonormal vectors u_j for $1 \le j \le k, j \ne i$ such that $\bigoplus_{1\le j\le k, j\ne i} M_jp(v_j) = \bigoplus_{1\le j\le k, j\ne i} N_ju_j$. Hence $N = N_1u_1 \oplus \cdots \oplus N_ku_k$ with $N_1 \cong M_1, \ldots, N_k \cong M_k$ scalar non-zero neutrices and $u_1, \ldots, u_k$ orthonormal vectors. ■

Theorem 2.14. *Let $k \in \mathbb{N}$ be standard. Let $N \subseteq \mathbb{R}^k$ be a neutrix. Assume $N = M_1v_1 \oplus \cdots \oplus M_kv_k$, where $M_1, \ldots, M_k$ are scalar non-zero neutrices and $v_1, \ldots, v_k$ are linearly independent vectors. Then $M_1, \ldots, M_k$ are unique up to isomorphism.*

Proof. The theorem follows from Theorem 2.13 and the Orthogonal decomposition theorem: The set of scalar neutrices of an orthogonal decomposition is uniquely determined. ■

In conclusion, the Orthogonal decomposition theorem for neutrices of axiomatic nonstandard analysis seems strong in the sense that, due to full saturation, there are no restrictions with respect to external cardinalities in the definition of the neutrix, and fine in the sense that uniqueness of the components is obtained with respect to equality. The Algebraic decomposition theorem implies rather easily the Orthogonal decomposition theorem with respect to existence, while the Orthogonal decomposition theorem implies rather easily the algebraic decomposition theorem with respect to uniqueness.

In addition, some care will be needed in attempts to prove the properties of neutrices of axiomatic nonstandard analysis by plain algebraic methods. In principle, they should remain within a sufficiently weak fragment of set theory, not including the power set-axiom or choice.

3. Thickness, width, and length of neutrices, decomposition in two dimensions

Definition 3.1. Let $k \in \mathbb{N}$, $k \geq 1$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix. We call N *square* in case there exists a scalar neutrix M such that $N = M^k$.

Definition 3.2. Let $k \in \mathbb{N}$, $k \geq 1$ be standard. The *thickness* of a neutrix $N \subset \mathbb{R}^k$ in the direction of a unit vector u is the scalar neutrix $T_u = \{x \in \mathbb{R} \,|\, xu \in N\}$. The thickness of a neutrix $N \subset \mathbb{R}^k$ in the direction of a non-zero vector r is the thickness in the direction of $u = r/\|r\|$.
The *width* W of N is defined by $W = \cap_{\|u\|=1} T_u$, and its *length* L by $L = \cup_{\|u\|=1} T_u$.

Consider the simple example $N = £ \times \oslash$. If $u = \begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix}$ with $\phi \simeq 0$, then $T_u = £$, and if $0 \lnapprox \phi \lnapprox \pi$, then $T_r = \oslash$. Hence $W = \oslash$ and $L = £$. For square neutrices, all thicknesses are equal.

The Orthogonal decomposition theorem in two dimensions acts as an initial step in the external induction leading to the proof of the general case, and is the main result of [26] as follows.

Theorem 3.3. *Let $N \subseteq \mathbb{R}^2$ be a neutrix of width W and length L. Then there are orthonormal vectors u_1 and u_2 such that $N = Lu_1 \oplus Wu_2$. Moreover, if there are scalar neutrices $M_1 \supseteq M_2$ and orthonormal vectors v_1, v_2 with $N = M_1v_1 \oplus M_2v_2$, it holds that $M_1 = L$ and $M_2 = W$.*

We give a new proof, based on the S-maximality of the ring £. The decomposition theorem is evident for square neutrices, and it is almost straightforward to prove the decomposition theorem for neutrices in $\mathbb{R}^2$ if their length is of the form $\lambda£$, with $\lambda \in \mathbb{R}$. Indeed, $N = \lambda£u \oplus Wv$ for any unit vector u such that $\lambda u \in N$, noting that any vector v perpendicular to u realizes the width W.

All other types of neutrices will be called *lengthy*:

Definition 3.4. Let $N \subseteq \mathbb{R}^2$ be a neutrix. We call N *lengthy* if it is not square, and if its length is not of the form $L = \lambda£$ for some $\lambda \in \mathbb{R}$, $\lambda > 0$.

Lengthy neutrices have the property that for every positive $\lambda \in L$ there exists $n \in N$ such that $\|n\|/\lambda \simeq \infty$, in particular every lengthy neutrix contains vectors of norm infinitely large with respect to every element of its width. Clearly, neutrices which are invariant under multiplication by an unlimited number are lengthy. In order to present some examples, we let $\epsilon > 0$ be infinitesimal. Neutrices with length $£\epsilon^{\not\infty}$ or $£e^{-@/\epsilon}$ are lengthy neutrices, for $\frac{1}{\epsilon} \cdot £\epsilon^{\not\infty} = £\epsilon^{\not\infty}$ and $\frac{1}{\epsilon} \cdot £e^{-@/\epsilon} = £e^{-@/\epsilon}$. Neutrices with length $H_\epsilon \equiv \cap\{[-f(\epsilon), f(\epsilon)] \,|\, f : (0,\infty) \to (0,\infty) \text{ is standard}\}$ or $G_\epsilon \equiv \cup\{[-f(\epsilon), f(\epsilon)] \,|\, f : (0,\infty) \to (0,\infty) \text{ is standard}\}$, are lengthy too, since obviously these scalar neutrices are also invariant under multiplication by $1/\epsilon$.

Up to a rescaling one may suppose that $W \subsetneqq \oslash$ and $L \supsetneqq £$. If this is the case, we call N *appropriately scaled*. It is now easy to see that the above mentioned short proof "from the inside" to prove the two dimensional decomposition theorem does not work for lengthy neutrices. Indeed, let u be a unit vector such that $\alpha u \in N$ for all limited $\alpha \in \mathbb{R}$. It may happen that $\lambda u \in N$ for some unlimited $\lambda \in L$, and then it is not true that $N = \lambda£u \oplus Wv$, with v a unit vector perpendicular to u.

If N is appropriately scaled, up to a rotation we may suppose that it is *appropriately oriented*, i.e.,

$$N \cap £ \times £ \subseteq £\begin{pmatrix}1 \\ 0\end{pmatrix} \oplus \oslash\begin{pmatrix}0 \\ 1\end{pmatrix}.$$

Once these convenient transformations having been carried out, we give the proof of the two-dimensional Orthogonal decomposition theorem. The most important step in the proof consists of establishing a direction which realizes the length of the neutrix, since the neutrix assumes its width in the direction perpendicular to it.

Proof of Theorem 3.3. By the above we need only to prove the case of lengthy neutrices, which we may suppose to be appropriately scaled and oriented. Define for $i \in L$, $i \simeq +\infty$

$$\alpha_i = \left\{ y \in \mathbb{R} \,\middle|\, i\begin{pmatrix}1\\ y\end{pmatrix} \in N \right\}. \tag{1}$$

It follows readily from geometric considerations and the fact that $tg\theta/\theta \simeq 1$ for infinitesimal polar angles that α_i is an external number for all $i \in L$, $i \simeq +\infty$, in fact it is of the form $\alpha_i = a_i + \frac{W}{i}$, with $a_i \simeq 0$. Put

$$F = \{\alpha_i \,|\, i \in L, i \simeq +\infty\}.$$

It follows from (1) that $\alpha_j \subseteq \alpha_i$ for all $i, j \in L$, $i, j \simeq +\infty$ such that $j \geq i$. This implies that F has the finite intersection property. By Theorem 2.4 $\cap F$ is not empty, in fact it is an external number of the form $\alpha = a + \cap_{i\in I} \frac{W}{i}$. Put $u_1 = \frac{1}{\sqrt{1+a^2}}\binom{1}{a}$ and $u_2 = \frac{1}{\sqrt{1+a^2}}\binom{-a}{1}$. Then u_1 and u_2 are orthonormal. The verification that $N = Lu_1 \oplus Wu_2$ is by straightforward linear algebra and geometry.

As for uniqueness, suppose $N = M_1 v_1 \oplus M_2 v_2$ for some orthonormal vectors v_1, v_2 and neutrices M_1, M_2 with $M_1 \supseteq M_2$. Clearly $M_1 = L$, for neither N does not contain vectors with norm larger than L, nor it is possible that all vectors of N have norms less than a given element of L. Necessarily $v_1 \simeq \pm u_1$. Then it is obvious that $v_2 \simeq \pm u_2$ and that $M_2 = T_{v_2} = W$. ■

4. Geometric properties of neutrices in $\mathbb{R}^k$

We start with a theorem which implies that, given a neutrix in $\mathbb{R}^k$, if we take an arbitrary orthonormal basis of $\mathbb{R}^k$, one of the directions chosen realizes its width. We start with some useful geometric properties of neutrices in $\mathbb{R}^2$.

Theorem 4.1 (*Sector Theorem*). *Let $N \subseteq \mathbb{R}^2$ be a neutrix and a and b two unit vectors so that the angle θ between a and b satisfies $0 \leq \theta \lessapprox \pi$. If c is a unit vector that makes an angle γ with a where $0 \leq \gamma \leq \theta$, then $T_c \geq \min(T_a, T_b)$.*

Proof. Suppose $\alpha \in T_a \subseteq T_b$ so $\alpha a, \alpha b \in N$. We show that $\alpha c \in N$. The line $\mathbb{R}c$ intersects the segment ab , and since $0 \leq \gamma \leq \theta \lessapprox \pi$, $\mu c = \lambda a + (1-\lambda)b$ for $0 \leq \lambda \leq 1$ and $0 \not\simeq \mu \leq 1$. The convex combination $\alpha\mu c = \lambda\alpha a + (1-\lambda)\alpha b \in N$ and $1/\mu$ is limited, so $\alpha c \in N$. ■

Theorem 4.2. *Let $N \subseteq \mathbb{R}^2$ be a neutrix of width W. There is an external interval of polar angles θ of unit vectors u, such that $T_u = W$ for all $\alpha \lessapprox \theta \lessapprox \alpha + \pi$.*

Proof. Let $a = \binom{1}{0}$ and $b = \binom{0}{1}$, and let $M = T_a \cap T_b$. We will apply the Sector Theorem to quadrants to prove that $M = W$, the width of N. The Sector Theorem shows that any unit vector c in the first quadrant satisfies $T_c \supseteq M$. For any unit vector u, the thickness satisfies $T_u = T_{-u}$, so every c in the third quadrant also satisfies $T_c \supseteq M$. If $a' = \binom{0}{1}$ and $b' = \binom{-1}{0}$, $M = T_{a'} \cap T_{b'} = T_b \cap T_{-a} = T_a \cap T_b$. Applying the Sector Theorem to a' and b' shows that any unit vector c' in either the second or fourth quadrant satisfies $T_{c'} \supseteq M$. By definition, the width $W = \cap_{\|u\|=1} T_u$ and we just showed that every $T_u \supseteq M$, so $W \supseteq M$. Clearly $W \subseteq M$, since M is an intersection of only two intersection terms making up W. Thus $W = M$. Also note that this shows that either $W = T_a$ or $W = T_b$ since $W = T_a \cap T_b = \min(T_a, T_b)$.
If N is square, every unit vector u has $W = T_u$, so suppose N is not square and there is a unit vector with $T_u \supset W$ where the polar angle of u is α. The argument above shows that for any orthonormal vectors a and b, $W = T_a$ or $W = T_b$, so if v is perpendicular to u we have $W = T_v$. If w is a unit vector at a polar angle β with $\alpha \lessapprox \beta \lessapprox \alpha + \pi$, we show by contradiction that $T_w = T_v = W$. Suppose $T_w \supset W$, since $T_u \supset W$, the Sector Theorem applied to u (or $-u$) and w with v in between, gives the contradiction that $T_v \supseteq T_u = T_w \neq W$. ■

Theorem 4.3. *Let $k \in \mathbb{N}$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix. Let $1 \leq j \leq k$, and let $u_1, \dots, u_j$ be orthonormal vectors. Then $T_v \supseteq \min_{1\leq i\leq j} T_{u_i}$ for any unit vector $v \in \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_j$.*

Proof. We use external induction on j. The property is evident if $j = 1$. Assume the property has been proved for $j-1$. Let $u_1, \dots, u_j$ be orthonormal, and $v \in \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_j$ be a unit vector. If $v \in \mathbb{R}u_j$, the property is obvious. If not, let $P(v)$ be the orthogonal projection of v on $\mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_{j-1}$. By the induction hypothesis $T_{P(v)} \supseteq \min_{1\leq i<j} T_{u_i}$. Notice that $v - P(v) \in \mathbb{R}u_j$. Because v is a linear combination of u_j and $P(v)$ and these vectors are orthogonal, it follows from Theorem 4.1 that $T_v \supseteq \min(T_{u_j}, T_{P(v)})$. We conclude that $T_v \supseteq \min_{1\leq i\leq j} T_{u_i}$. ■

Corollary 4.4. *Let $k \in \mathbb{N}$, $k \geq 1$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix with width W. Then there exists a unit vector u such that $T_u = W$.*

Proof. Apply Theorem 4.3 to an orthonormal basis $u_1, \dots, u_k$. ■

Next we introduce some notions, which are slight adaptations of notions of common linear algebra and euclidean geometry.

Definition 4.5. Let $k \in \mathbb{N}$, $k \geq 2$ be standard, $N \subseteq \mathbb{R}^k$ be a neutrix and $x, y \in \mathbb{R}^k$. We call x and y *nearly orthonormal* if $\| x \| = \| y \| = 1$ and $\langle x, y\rangle \simeq 0$. We call x and y *nearly orthogonal* if $\frac{x}{\|x\|}$ and $\frac{y}{\|y\|}$ are nearly orthonormal.

For example, if $\epsilon \simeq 0$, the vectors $\binom{1}{0}$ and $\binom{\epsilon}{\sqrt{1-\epsilon^2}}$ are nearly orthonormal and the vectors $\binom{1/\epsilon}{0}$ and $\binom{\epsilon}{1}$ are nearly orthogonal.

Definition 4.6. Let $k \in \mathbb{N}$, $k \geq 2$ be standard, $N \subseteq \mathbb{R}^k$ be a non-square neutrix and $x, y \in \mathbb{R}^k$ be non-zero vectors. We call a line $\mathbb{R}y$ *nearly parallel* with N if $T_y \supset W$. We call x *nearly normal* to N if x is nearly orthogonal to all non-zero vectors z such that $\mathbb{R}z$ is nearly parallel with N.

As an example, consider the neutrix $N \equiv £ \times \oslash \subseteq \mathbb{R}^2$. The vector $\binom{0}{1}$ is nearly normal to N and all vectors of the form $\binom{\alpha}{1+\beta}$ with $\alpha, \beta \simeq 0$ are also nearly normal to N.

We motivate the terminology of nearly-parallel and nearly-normal of Definition 4.6 for non-square neutrices N in two dimensions. By Theorem 3.3 $N = Lu_1 \oplus Wu_2$, where u_1 and u_2 are orthonormal vectors, L is the length of N and W is the width of N. As a consequence of Theorem 4.2 a unit vector v_1 such that $T_{v_1} \supset W$ satisfies $v_1 \simeq u_1$ or $v_1 \simeq -u_1$. Then the polar angle between the lines $\mathbb{R}v_1$ and $\mathbb{R}u_1$ is infinitesimal, and because the neutrix N realizes its length in the direction u_1 it is natural to call the line $\mathbb{R}v_1$ nearly parallel with N.

Definition 4.7. Let $q, k \in \mathbb{N}$ be standard with $1 \leq q \leq k$. Let $x \in \mathbb{R}^k$ be a unit vector and $u_1, \ldots, u_q$ be orthonormal. Assume there are $\alpha_1, \ldots \alpha_q \in \mathbb{R}$ with $|\alpha_1|, \ldots, |\alpha_q| \leq 1$ such that

$$x \simeq \alpha_1 u_1 + \cdots + \alpha_q u_q.$$

Then x is called *nearly generated* by $u_1, \ldots, u_q$.

Definition 4.8. Let $k \in \mathbb{N}$, $k \geq 2$ be standard and $N \subseteq \mathbb{R}^k$ be a non-square neutrix with width W. Let $q \in \mathbb{N}$ be maximal such that there are orthonormal vectors $u_1, \ldots, u_q$ with $T_{u_1}, \ldots, T_{u_q} \supset W$. Put $Y = \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_q$. Then Y will be called a *nonminimal component* for N, and q the *nonminimal dimension* of N. Let Z be the orthogonal complement of Y. Then Z will be called the *corresponding minimal component* for N, and $k - q$ the *minimal dimension* of N.

By Corollary 4.4 one has $q < k$, hence also $k-q > 0$. As an example, for a non-square neutrix N in $\mathbb{R}^2$ by definition any line nearly parallel with N is a nonminimal component. Also, in $\mathbb{R}^3$ any nearly horizontal plane (i.e. a plane with normal vector of the form $(\alpha, \beta, 1+\gamma)$, with $\alpha \simeq \beta \simeq \gamma \simeq 0$) is a nonminimal component for the neutrix $M = \mathbb{R}\begin{pmatrix}1\\0\\0\end{pmatrix} \oplus £ \begin{pmatrix}0\\1\\0\end{pmatrix} \oplus \oslash \begin{pmatrix}0\\0\\1\end{pmatrix}$.

Theorem 4.11 is a kind of generalization of Theorem 4.2 and states that non-square neutrices are "thin", for any unit vector corresponding to a nonminimal direction is nearly generated by an orthonormal basis of a nonminimal component. We present first some elementary properties of near-parallelness and near-orthogonality.

Proposition 4.9. *Let $k \in \mathbb{N}$, $k \geq 2$ be standard and $N \subseteq \mathbb{R}^k$ be a non-square neutrix with width W. Let Y be a nonminimal component for N, and P be the orthogonal projection on Y.*

1. *Let x be a unit vector such that $T_x > W$. Then $P(x) \simeq x$.*
2. *Let v be a unit vector nearly normal to N. Then $P(v) \simeq 0$.*

Proof. 1. If $x \in Y$, one has $P(x) = x$. If not, the vector $x - P(x)$ is orthogonal to Y, so $T_{x-P(x)} = W$. It follows from Theorem 4.3 that $T_{P(x)} > W$. Then the result follows if we apply Theorem 4.2 to the plane $\mathbb{R}x \oplus \mathbb{R}(x - P(x))$.

2. Let $u_1, \ldots, u_k$ be an orthonormal basis of $\mathbb{R}^k$ such that $u_1, \ldots, u_q$ is an orthonormal basis of Y and $u_{q+1}, \ldots, u_k$ is an orthonormal basis of the corresponding minimal component. Then $\langle v, u_1 \rangle \simeq \cdots \simeq \langle v, u_q \rangle \simeq 0$, so

$$P(v) = \langle v, u_1 \rangle u_1 + \cdots + \langle v, u_q \rangle u_q \simeq 0. \quad \blacksquare$$

Proposition 4.10. *Let $q, k \in \mathbb{N}$ be standard, with $1 \leq q \leq k$ and $v_1, \ldots, v_q$ be nearly orthonormal vectors in $\mathbb{R}^k$. Then $v_1, \ldots, v_q$ are linearly independent.*

Proof. Let $\alpha_1, \ldots, \alpha_k$ be such that

$$\alpha_1 v_1 + \cdots + \alpha_q v_q = 0.$$

By changing the order, if necessary, we may assume that $|\alpha_1|$ is maximal. There exist $\epsilon_1 \simeq \cdots \simeq \epsilon_q \simeq 0$ such that

$$\langle v_1, \alpha_1 v_1 + \cdots + \alpha_q v_q \rangle = \alpha_1(1 + \epsilon_1) + \alpha_2 \epsilon_2 + \cdots + \alpha_q \epsilon_q = 0.$$

Then it follows from the maximality of $|\alpha_1|$ that there exists $\epsilon \simeq 0$ such that $\alpha_2 \epsilon_2 + \cdots + \alpha_q \epsilon_q = \epsilon \alpha_1$. So

$$\alpha_1(1 + \epsilon_1 + \epsilon) = 0,$$

which implies that $\alpha_1 = 0$. Again by the maximality of $|\alpha_1|$ we conclude that $\alpha_2 = \cdots = \alpha_q = 0$. Hence $v_1, \ldots, v_q$ are linearly independent. $\blacksquare$

Theorem 4.11. *Let $q, k \in \mathbb{N}$ be standard, with $1 \leq q \leq k$ and $k \geq 2$, and $N \subseteq \mathbb{R}^k$ be a nonsquare neutrix. Then q is the nonminimal dimension of N if and only if there are orthonormal vectors $v_1, \ldots, v_q$ such that every unit vector x such that $T_x > W$ is nearly generated by $v_1, \ldots, v_q$.*

Proof. Let $v_1, \ldots, v_q$ be unit vectors with $T_{v_1}, \ldots, T_{v_q} \supset W$ such that $Y = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_q$ is a nonminimal component for N. Let P be the orthogonal projection on Y. Let x be a unit vector such that $T_x \supset W$. Because $\|P(x)\| \leq \|x\| = 1$, there exist $\alpha_1, \ldots, \alpha_q \in \mathbb{R}$, with $|\alpha_1|, \ldots, |\alpha_q| \leq 1$, such that

$$P(x) = \alpha_1 v_1 + \cdots + \alpha_q v_q.$$

By Proposition 4.9 one has $x \simeq P(x)$. Hence x is nearly generated by $v_1, \ldots, v_q$.

Conversely, let $j \in \mathbb{N}$ and $v_1, \ldots, v_j$ be orthonormal vectors such that every unit vector x such that $T_x \supset W$ is nearly generated by $v_1, \ldots, v_j$. Let $V = \mathbb{R}v_1 \oplus \cdots \oplus \mathbb{R}v_j$. Let $u_1, \ldots, u_q$ be unit vectors with $T_{u_1}, \ldots, T_{u_q} \supset W$ such that $U = \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_q$ is a nonminimal component for N. We prove that $j = q$. Let Q be the orthogonal projection on U. By Proposition 4.9 one has $Q(v_i) \simeq v_i$ for all i with $1 \leq i \leq j$, so by Proposition 4.10 the dimension of $Q(V)$ is equal to j. Because $Q(V) \subseteq U$, we derive that $j \leq q$. Suppose $j < q$. Then there exists a unit vector $u \in U$ orthogonal to $Q(V)$. Now $T_u \supset W$ by Theorem 4.3, so there are $\alpha_1, \ldots, \alpha_j \in \mathbb{R}$ with $|\alpha_1|, \ldots, |\alpha_j| \leq 1$ such that $u \simeq \alpha_1 v_1 + \cdots + \alpha_j v_j$. Let i be such that $1 \leq i \leq j$. Then

$$\alpha_i \simeq \langle u, v_i \rangle \simeq \langle u, Q(v_i) \rangle = 0.$$

So $\|u\| \simeq 0$, hence u cannot be a unit vector. We conclude that $j = q$, the nonminimal dimension of N. ■

5. Proof of the Orthogonal decomposition theorem

As was the case for neutrices in two dimensions, once we identify a direction which realizes the length, the existence of the decomposition is fairly easy to prove. As regards to the nontrivial case of a nonsquare neutrix N we use orthogonal projection on a non-minimal component and the two-dimensional orthogonal decomposition theorem to show that N realizes its length in some direction. An important step in the proof is given by the next result, which establishes the shape of the intersection of N with a plane which cuts both a nonminimal component and its corresponding minimal component.

Theorem 5.1. *Let $k \in \mathbb{N}$, $k \geq 2$ be standard and $N \subseteq \mathbb{R}^k$ be a nonsquare neutrix with width W. Let Y be a nonminimal component for N and Z be the corresponding minimal component. Let $y \in Y$, $z \in Z$ be orthonormal vectors. Let P be the orthogonal projection on $\mathbb{R}y$. Let $M \subseteq \mathbb{R}$ be such that*

$$P(N \cap \mathbb{R}y \oplus \mathbb{R}z) = My.$$

Then M is a neutrix, and there are orthonormal vectors $u \simeq y$ and $v \simeq z$ such that

$$N \cap \mathbb{R}y \oplus \mathbb{R}z = Mu \oplus Wv.$$

Proof. Because P is a linear mapping, the set M is a neutrix. Because $T_z = W$ the width of $N \cap \mathbb{R}y \oplus \mathbb{R}z$ is also equal to W. By the two-dimensional decomposition-theorem there are orthonormal vectors u, v and a neutrix $K \subseteq \mathbb{R}$ such that

$$N \cap \mathbb{R}y \oplus \mathbb{R}z = Ku \oplus Wv.$$

Now $K \supset W$, or else $M = W$, a contradiction. Then it follows from Theorem 4.2 that we may assume that $u \simeq y$, hence also that $v \simeq z$. Because $P(u) \simeq u$, we have $P(Ku) = KP(u) = Ky$, so $K \subseteq M$. Conversely, let $m \in M$, and $n \in N \cap \mathbb{R}y \oplus \mathbb{R}z$ be such that $P(n) = my$. Then, noting that K is the length of $N \cap \mathbb{R}y \oplus \mathbb{R}z$,

$$|m| \leq \|n\| \in K.$$

So $M \subseteq K$. We conclude that $K = M$, which finishes the proof. ■

Theorem 5.2. *Let $k \in \mathbb{N}$, $k \geq 1$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix with length L. Then there is a unit vector u such that $T_u = L$.*

Proof. We use external induction. The case $k = 1$ is obvious, and the case $k = 2$ is contained in the two-dimensional orthogonal decomposition theorem. Suppose the theorem holds for neutrices in $\mathbb{R}^{k-1}$. Let $N \subseteq \mathbb{R}^k$ be a neutrix with length L and width W. If N is square, one may take any unit vector. If not, let Y be a non-minimal component, and Z be its corresponding minimal component. Let P be the orthogonal projection on Y. Then $P(N)$ is a neutrix within a subspace with dimension less or equal to $k - 1$. Then there exists a unit vector $y \in Y$ such that T_y is the length of $P(N)$. Let $z \in Z$ be a unit vector, and consider the plane $N \cap \mathbb{R}y \oplus \mathbb{R}z$. Because $P(\mathbb{R}z) = \{0\}$, we have

$$T_y y = P(N \cap \mathbb{R}y) = P(N \cap \mathbb{R}y \oplus \mathbb{R}z).$$

By Theorem 5.1 there exist orthonormal vectors $u \simeq y$ and $v \simeq z$ such that

$$N \cap \mathbb{R}y \oplus \mathbb{R}z = T_y u \oplus Wv.$$

We prove that $T_y = L$. Because $\|P(n)\| \leq \|n\|$ for any $n \in N$, we have $T_y \subseteq L$. Let $\lambda x \in N$ with $\lambda > W$ and x a unit vector. By Proposition 4.9 it holds that $P(x) \simeq x$, so $P(T_x x) = T_{P(x)} P(x) = T_x P(x)$, from which we derive that $T_x \subseteq T_y$. Hence $L \subseteq T_y$. We conclude that $L = T_y$, which means that $N \cap \mathbb{R}u = Lu$. ■

Theorem 5.3. *Let $k \in \mathbb{N}$, $k \geq 1$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix with length L and width W. Then there are scalar neutrices $N_1, \ldots, N_k$ with $L = N_1 \supseteq \cdots \supseteq N_k = W$ and orthonormal vectors $u_1, \ldots, u_k$ such that*

$$N = N_1 u_1 \oplus \cdots \oplus N_k u_k.$$

Proof. We use external induction. The case $k = 1$ is obvious, and the two-dimensional orthogonal decomposition theorem concerns the case $k = 2$. If N is square, the theorem is also obvious. Suppose the theorem holds for neutrices in $\mathbb{R}^{k-1}$, and let $N \subseteq \mathbb{R}^k$ be a nonsquare neutrix with length L and width W. By Theorem 5.2 there is a unit vector u_1 such that $N \cap \mathbb{R}u_1 = Lu_1$. Put $N_1 = L$ and let $U = \mathbb{R}u_1^{\perp}$. By the induction hypothesis there are neutrices $N_2, \dots, N_k \subseteq \mathbb{R}$ with $N_2 \supseteq \cdots \supseteq N_k = W$ and orthonormal vectors $u_2, \dots, u_k$ such that

$$N \cap U = N_2 u_2 \oplus \cdots \oplus N_k u_k.$$

Because N is a group,

$$N_1 u_1 \oplus N_2 u_2 \oplus \cdots \oplus N_k u_k \subseteq N.$$

Conversely, let $n \in N$. Let P be the orthogonal projection on $\mathbb{R}u_1$. Now $\|n\| \in L = N_1$, so

$$\|P(n)\| \le \|n\| \in N_1.$$

This implies that $P(n) \in N_1 u_1 \subseteq N$. Because N is a group, also $n - P(n) \in N$. So $n - P(n) \in N \cap U$, hence

$$n \in N_1 u_1 \oplus N_2 u_2 \oplus \cdots \oplus N_k u_k.$$

This means that $N \subseteq N_1 u_1 \oplus N_2 u_2 \oplus \cdots \oplus N_k u_k$. We conclude that $N = N_1 u_1 \oplus \cdots \oplus N_k u_k$. This proves the theorem. ■

Given the decomposition, we can easily recognize a nonminimal and a minimal component. This is done in the next theorem, which will be used in proving the uniqueness of the decomposition.

Theorem 5.4. *Let $k \in \mathbb{N}$, $k \ge 2$ be standard and $N \subseteq \mathbb{R}^k$ be a nonsquare neutrix with width W. Let $u_1, \dots, u_k$ be orthonormal vectors and $N_1, \dots, N_k$ be scalar neutrices with $N_1 \supseteq \cdots \supseteq N_k = W$ such that*

$$N = N_1 u_1 \oplus \cdots \oplus N_k u_k.$$

Let $q < k$ be such that $N_q \supset W$ and $N_{q+1} = W$. Then q is the non-minimal dimension of N, the subspace $\mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_q$ a nonminimal component, and the subspace $\mathbb{R}u_{q+1} \oplus \cdots \oplus \mathbb{R}u_k$ a corresponding minimal component.

Proof. Up to a rescaling we may assume that $W \subseteq \oslash$. Consider a direction with nonminimal thickness T, which again up to a rescaling we may assume to satisfy $T \supseteq £$. Let then x be a unit vector such that $T = T_x$. Then $x \in N$. Let $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ with $|\alpha_1|, \dots, |\alpha_k| \le 1$ be such that

$$x = \alpha_1 u_1 + \cdots + \alpha_q u_q + \alpha_{q+1} u_{q+1} + \cdots + \alpha_k u_k$$

then $\alpha_{q+1}, \dots, \alpha_k \in W$, so $\alpha_{q+1} \simeq \cdots \simeq \alpha_k \simeq 0$. Hence $x \simeq \alpha_1 u_1 + \cdots + \alpha_q u_q$, which means that x is nearly generated by $u_1, \dots, u_q$. By Theorem 4.11 the nonminimal dimension of N is equal to q . Because $T_{u_1}, \dots, T_{u_q} \supset W$ and $T_{u_{q+1}}, \dots, T_{u_k} = W$, the subspace $\mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_q$ is a nonminimal component for N, and the subspace $\mathbb{R}u_{q+1} \oplus \cdots \oplus \mathbb{R}u_k$ is the corresponding minimal component. ■

In proving the uniqueness of the decomposition we use also projection on a non-minimal component. The projection will deform an orthonormal basis into a nearly-orthonormal system, that on behalf of Proposition 4.10 we know to be independent. Theorem 2.13 states that we can recover from this system a true orthonormal system, possibly modifying the scalar neutrices by isomorphisms. Next theorem states that the scalar neutrices of the orthonormal system remain the same.

Theorem 5.5. *Let $k \in \mathbb{N}$, $k \ge 1$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix. Assume there are scalar neutrices $N_1, \dots, N_k$ with $N_1 \supseteq \cdots \supseteq N_k$ and nearly orthonormal vectors $v_1, \dots, v_k$ such that*

$$N = N_1 v_1 + \cdots + N_k v_k.$$

Then there are orthonormal vectors $u_1, \dots, u_k$ such that

$$N = N_1 u_1 \oplus \cdots \oplus N_k u_k.$$

Proof. Because $v_1, \dots, v_k$ are independent, we may apply the common recursive Gram–Schmidt orthogonalization process to obtain orthogonal vectors w_j, $1 \le j \le k$ of the form

$$w_j = v_j - \frac{\langle v_j, w_{j-1}\rangle}{\|w_{j-1}\|^2} w_{j-1} - \cdots - \frac{\langle v_j, w_1\rangle}{\|w_1\|^2} w_1.$$

We prove by external induction that there are $\beta_i \simeq \gamma_i \simeq 0$, with $1 \le i < j$, such that both

$$w_j = v_j + \beta_{j-1} v_{j-1} + \cdots + \beta_1 v_1 \tag{2}$$

and

$$v_j = w_j + \gamma_{j-1} w_{j-1} + \cdots + \gamma_1 w_1. \tag{3}$$

Clearly $w_1 = v_1$. Suppose that $i \leq j$ and that (2) and (3) hold for $i - 1$. Then for $h < i$

$$\frac{\langle v_i, w_h\rangle}{\|w_h\|^2} \simeq \langle v_i, w_h\rangle \simeq 0,$$

so

$$\begin{aligned} w_i &\in v_i - \oslash(v_{i-1} + \oslash v_{i-2} + \cdots + \oslash v_1) - \cdots - \oslash v_1 \\ &= v_i + \oslash v_{i-1} + \oslash v_{i-2} + \cdots + \oslash v_1. \end{aligned}$$

Moreover

$$\begin{aligned} v_i &= w_i + \frac{\langle v_i, w_{i-1}\rangle}{\|w_{i-1}\|^2} w_{i-1} + \cdots + \frac{\langle v_i, w_1\rangle}{\|w_1\|^2} w_1 \\ &\in w_i + \oslash w_{i-1} + \cdots + \oslash w_1. \end{aligned}$$

Put $u_j = w_j / \left\|w_j\right\|$ for $1 \leq j \leq k$. Then $u_1, \ldots, u_k$ are orthonormal, and because $\left\|w_j\right\| \simeq \left\|u_j\right\| = 1$ for $1 \leq j \leq k$, we derive from (2) and (3) that for every j with $1 \leq j \leq k$ there are $\delta_i \simeq \epsilon_i \simeq 0$, with $1 \leq i < j$ such that

$$u_j = v_j + \delta_{j-1} v_{j-1} + \cdots + \delta_1 v_1 \tag{4}$$

and

$$v_j = u_j + \epsilon_{j-1} u_{j-1} + \cdots + \epsilon_1 u_1. \tag{5}$$

In order to prove the remaining part of the theorem we put for $1 \leq j \leq k$

$$\begin{aligned} V_j &= N_1 v_1 + \cdots + N_j v_j \\ U_j &= N_1 u_1 \oplus \cdots \oplus N_j u_j, \end{aligned}$$

and show that $U_j = V_j$ by external induction. Clearly $N_1 u_1 = N_1 v_1$. Let $x \in U_j$. Then there are $n_j \in N_j$ and $y \in U_{j-1}$ such that

$$x = y + n_j u_j.$$

Now $y \in V_{j-1}$ by the induction hypothesis. Applying (4), and the fact that $N_j \subseteq N_i$ for all i with $1 \leq i < j$ we obtain

$$\begin{aligned} x &\in N_1 v_1 + \cdots + N_{j-1} v_{j-1} + N_j v_j + \oslash N_{j-1} v_{j-1} + \cdots + \oslash N_1 v_1 \\ &= N_1 v_1 + \cdots + N_{j-1} v_{j-1} + N_j v_j \\ &= V_j. \end{aligned}$$

So $U_j \subseteq V_j$. The converse is proved in the same way, now applying (5). We conclude that $U_j = V_j$. We finish the proof by applying this equality to $j = k$. ■

Theorem 5.6. *Let $k \in \mathbb{N}$, $k \geq 1$ be standard and $N \subseteq \mathbb{R}^k$ be a neutrix. Assume there are neutrices $N_1, \ldots, N_k, M_1, \ldots, M_k \subseteq \mathbb{R}$ with $N_1 \supseteq \cdots \supseteq N_k$ and $M_1 \supseteq \cdots \supseteq M_k$, and orthonormal vectors $u_1, \ldots, u_k$ and $t_1, \ldots, t_k$ such that*

$$N = N_1 u_1 \oplus \cdots \oplus N_k u_k = M_1 t_1 \oplus \cdots \oplus M_k t_k.$$

Then $M_i = N_i$ for all i with $1 \leq i \leq k$.

Proof. The length of a neutrix will always be denoted by L, and its width by W. We use external induction in k. If $k = 1$, $N_1 = M_1 = L$, by (for instance) Theorem 5.3. Assume the uniqueness is proved for all neutrices within linear spaces of dimension less or equal to $k - 1$. Let $N \subseteq \mathbb{R}^k$ be a neutrix. If N is square, we have $N_1 = \cdots = N_k = M_1 = \cdots = M_k = L$. If not, by Theorem 5.4 there exists $q < k$ such that both $U \equiv \mathbb{R}u_1 \oplus \cdots \oplus \mathbb{R}u_q$ and $X \equiv \mathbb{R}t_1 \oplus \cdots \oplus \mathbb{R}t_q$ are nonminimal components for N and

$$N_{q+1} = \cdots = N_k = M_{q+1} = \cdots = M_k = W. \tag{6}$$

Let P be the orthogonal projection on U. Then $P(N) = N_1 u_1 \oplus \cdots \oplus N_q u_q$. We show that there exist orthonormal vectors $v_1, \ldots, v_q$ such that also $P(N) = M_1 v_1 \oplus \cdots \oplus M_q v_q$.

By Proposition 4.9.1 we have $P(t_i) \simeq t_i$ for $1 \leq i \leq q$. This implies that $P(t_1), \ldots, P(t_q)$ are nearly orthonormal. By Theorem 5.5 there is an orthonormal basis $v_1, \ldots, v_q$ of Y with $v_i \simeq t_i$ for $1 \leq i \leq q$ and

$$Q \equiv M_1 P(t_1) + \cdots + M_q P(t_q) = M_1 v_1 \oplus \cdots \oplus M_q v_q.$$

We show that

$$P(N) = Q.$$

By linearity of P one has $Q \subseteq P(N)$. Conversely, let Y be the minimal component for N corresponding to X. Let j be such that $q + 1 \leq j \leq k$. Applying Proposition 4.9.2 we see that there are $\epsilon_1, \ldots, \epsilon_q \simeq 0$ such that

$$P(t_j) = \epsilon_1 v_1 + \cdots + \epsilon_q v_q.$$

So

$$\begin{aligned} P(Wt_j) &\subseteq Wv_1 \oplus \cdots \oplus Wv_q \\ &\subseteq M_1v_1 \oplus \cdots \oplus M_qv_q \\ &= M_1P(t_1) + \cdots + M_qP(t_q) \\ &= Q. \end{aligned}$$

By linearity, also $P(Y) \subseteq Q$. Again by linearity

$$P(N) = P(X \oplus Y) = P(X) + P(Y) \subseteq Q + Q = Q.$$

We conclude that $P(N) = Q$. From this it follows that

$$N_1u_1 \oplus \cdots \oplus N_qu_q = M_1v_1 \oplus \cdots \oplus M_qv_q.$$

By the induction hypothesis it holds that $N_i = M_i$ for all i with $1 \le i \le q$. Together with formula (6) this completes the proof. ■

Combining Theorems 5.3 and 5.6 we obtain the Orthogonal decomposition Theorem 1.2.

6. On neutrices in $\mathbb{R}^\omega$ with $\omega \in \mathbb{R}$ unlimited

We show that the Orthogonal decomposition theorem is not valid in the space $\mathbb{R}^\omega$ with $\omega \in \mathbb{R}$ unlimited.

Theorem 6.1. *Let $\omega \in \mathbb{N}$ be unlimited. Let*

$$N = \{x \in \mathbb{R}^\omega \,|\, \|x\| \text{ is limited}\}.$$

Then N is a neutrix and there do not exist neutrices $N_1, \dots, N_\omega \subseteq \mathbb{R}$ and orthonormal vectors $u_1, \dots, u_\omega$ such that $N = N_1u_1 \oplus \cdots \oplus N_\omega u_\omega$.

Proof. Clearly $£N = N$. To show that N is convex, let $x, y \in N$ and $0 \le \lambda \le 1$. Then

$$\|\lambda x + (1-\lambda)y\| \le \lambda\,\|x\| + (1-\lambda)\,\|y\| \in £.$$

We conclude that N is a neutrix. Suppose there are orthonormal vectors $u_1, \dots, u_\omega$ and neutrices $N_1, \dots, N_\omega \subseteq \mathbb{R}$ such that $N = N_1u_1 \oplus \cdots \oplus N_\omega u_\omega$. Let i be such that $1 \le i \le \omega$. Clearly $T_{u_i} = £$, so $N_i = £$. So

$$N = £u_1 \oplus \cdots \oplus £u_\omega.$$

However $u_1 + \cdots + u_\omega \in £u_1 \oplus \cdots \oplus £u_\omega$, but $\|u_1 + \cdots + u_\omega\| = \sqrt{\omega} > £$, a contradiction. This proves the theorem. ■

In the same manner one proves that, for unlimited $\omega \in \mathbb{N}$, the external set $\{x \in \mathbb{R}^\omega \,|\, \|x\| \text{ is infinitesimal}\}$ is a neutrix, which is not of the form $\oslash u_1 \oplus \cdots \oplus \oslash u_\omega$ for some orthonormal basis $u_1, \dots, u_\omega$. Indeed, $\left\|\frac{1}{\sqrt{\omega}}u_1 + \cdots + \frac{1}{\sqrt{\omega}}u_\omega\right\| = 1 \gtrapprox 0$. On the contrary,

$$\left\{x \in \mathbb{R}^\omega \,\middle|\, \|x\| = £e^{-@\omega}\right\} = £e^{-@\omega}u_1 \oplus \cdots \oplus £e^{-@\omega}u_\omega$$

for every orthonormal basis $u_1, \dots, u_\omega$. This follows easily from the fact that $\sqrt{\omega}£e^{-@\omega} = £e^{-@\omega + \log\omega/2} = £e^{-@\omega}$.

7. A domain of approximation invariant under a two-dimensional neutrix

We present an example of calculations with neutrices in two variables. For convenience we use complex analysis. We extend the notion of *neutrix* to a convex subgroup of $\mathbb{C}$, and in fact such a neutrix N may be identified with a neutrix of $\mathbb{R}^2$. If N is of the form $N = \{x + iy \,|\, x \in N_1, y \in N_2\}$, where N_1, N_2 are scalar neutrices, we write $N = N_1 \oplus N_2 i$.

We study the approximation of the complex function e^z by the Euler formula $e_\omega(z) = \left(1 + \frac{z}{\omega}\right)^\omega$ for unlimited natural numbers ω. Let the external set $H \subseteq \mathbb{R}$ be defined by

$$H = \left\{z \,\middle|\, e_\omega(z) \simeq e^z\right\}.$$

Note that H is defined by a halic formula. We determine the shape of H and the external set of translations V leaving H invariant. The external set H is a mathematical model for the intuitive notion of a domain where the Euler formula may be considered as a good approximation of the exponential function. Similar sets, in one dimension, were determined in [22,23], for approximations by Taylor polynomials.

We will show that

$$H = H_1 \cup H_2,$$

where H_1 is the "open" disk centered in $-\omega$ with radius $\omega + £$, i.e.

$$H_1 = \{z \in \mathbb{C} \,|\, |z+\omega| - \omega \simeq -\infty\}, \tag{7}$$

and H_2 is a "bubble" on the right side of the disk, of the form

$$H_2 = \left\{x + iy \,\middle|\, y \in \oslash\sqrt{\omega}\mathrm{e}^{-x/2},\ -\infty \lessapprox x,\ x - \log\omega - 2\log\log\omega \simeq -\infty\right\}. \tag{8}$$

By inspection, H contains the neutrix £ ⊕ £i. This also follows from the nonstandard characterization of the fact that $e_n(z) \longrightarrow \mathrm{e}^z$ uniformly on every compact set. The Fehrele principle implies that H cannot be equal to the neutrix £ ⊕ £i, which is an external set defined by a galactic formula. So $H \supset$ £ ⊕ £i and in proving (7) and (8) we need only to consider the case $|z| \simeq +\infty$.

In order to determine H_1, notice that for $\mathrm{Re}\, z \simeq -\infty$ we have $\mathrm{e}^z \simeq 0$, so

$$H_1 = \left\{z \,\middle|\, \mathrm{Re}\, z \simeq -\infty,\ \left(1 + \frac{z}{\omega}\right)^{\omega} \simeq 0\right\}.$$

Observe that $@^{1/\omega} = \exp\frac{\text{£}}{\omega} = 1 + \frac{\text{£}}{\omega}$. Then

$$\left|1 + \frac{z}{\omega}\right|^{\omega} \in @ \Leftrightarrow \left(\frac{|\omega + z|}{\omega}\right)^{\omega} \in @ \Leftrightarrow \frac{|\omega + z|}{\omega} \in 1 + \frac{\text{£}}{\omega} \Leftrightarrow |\omega + z| - \omega \in \text{£}.$$

Hence

$$\left(1 + \frac{z}{\omega}\right)^{\omega} \simeq 0 \Leftrightarrow \left|1 + \frac{z}{\omega}\right|^{\omega} \simeq 0 \Leftrightarrow |\omega + z| - \omega \simeq -\infty.$$

This implies (7).

In order to determine H_2, by analyzing the order of magnitude of some of the involved quantities, we will successively reduce the domains where we have to look for solutions, and thus obtain some simplifications, applying Taylor expansions. Notice first that $|\mathrm{e}^z| \gtrapprox 0$ for $\mathrm{Re}\, z \gtrapprox -\infty$. One has

$$\exp z - \left(1 + \frac{z}{\omega}\right)^{\omega} = \exp z\left(1 - \exp\left(\omega\left(\log\left(1 + \frac{z}{\omega}\right) - \frac{z}{\omega}\right)\right)\right).$$

If $\left|\frac{z}{\omega}\right| \gtrapprox 0$, also $\left|\log\left(1 + \frac{z}{\omega}\right) - \frac{z}{\omega}\right| \gtrapprox 0$, so we need only to consider the case $z/\omega \simeq 0$, and then

$$\exp z\left(1 - \exp\left(\omega\left(\log\left(1 + \frac{z}{\omega}\right) - \frac{z}{\omega}\right)\right)\right) \in \exp z\left(1 - \exp\left(-(1 + \oslash)\frac{z^2}{2\omega}\right)\right).$$

Since $\mathrm{Re}\, z$ is not negative unlimited, $\exp z\left(1 - \exp\left(-(1 + \oslash)\frac{z^2}{2\omega}\right)\right)$ can only be infinitesimal if $1 - \exp\left(-\frac{z^2}{2\omega}\right)$ is infinitesimal. This implies the sharper estimate $z/\sqrt{\omega} \simeq 0$. Then we have

$$\exp z\left(1 - \exp\left(-(1 + \oslash)\frac{z^2}{2\omega}\right)\right) = (1 + \oslash)\frac{z^2}{2\omega}\exp z.$$

Hence it suffices to determine $\left\{z \,\middle|\, \frac{z^2}{\omega}\exp z \simeq 0\right\}$, or alternatively $\left\{z \,\middle|\, \frac{\left|z^2 \exp z\right|}{\omega} \simeq 0\right\}$. Otherwise said, we have to determine all real numbers x and y such that

$$\frac{\left(x^2 + y^2\right)\mathrm{e}^x}{\omega} \simeq 0. \tag{9}$$

We claim that if x satisfies $\frac{x^2\mathrm{e}^x}{\omega} \simeq 0$ we always have solutions of the form $x + iy$, where x/y is unlimited. Indeed, for all limited c we have

$$\frac{\left(x^2 + c^2x^2\right)\mathrm{e}^x}{\omega} = \frac{\left(1 + c^2\right)x^2\mathrm{e}^x}{\omega} \simeq 0,$$

and we conclude by the Fehrele principle. Then instead of (9), we may solve for

$$\frac{y^2\mathrm{e}^x}{\omega} \simeq 0.$$

Since numbers are infinitesimal if and only if their square roots are infinitesimal, we find for y

$$y \in \oslash\sqrt{\omega}\mathrm{e}^{-x/2}. \tag{10}$$

Such solutions y exist whenever $\frac{x^2\mathrm{e}^x}{\omega} \simeq 0$, or

$$x + 2\log x - \log\omega \simeq -\infty. \tag{11}$$

The Eqs. (10) and (11) imply (8).

The shape of H being determined, we consider now the set of translations which leave H invariant. It follows directly from (8) that the set of translations which leave H_1 invariant is equal to £ ⊕ £i. Because

$$\oslash\sqrt{\omega}\mathrm{e}^{-(x+\text{£})/2} = \oslash\sqrt{\omega}\mathrm{e}^{-x/2},$$

the set of horizontal translations which leave H_2 invariant contains £. For fixed x, the set $\oslash\sqrt{\omega}\mathrm{e}^{-x/2}$ is a neutrix, which contains £ strictly, so the set of vertical translations which leave H_2 invariant certainly contains £. Hence the set of translations which leaves H_2 invariant contains £ ⊕ £i. We conclude that the set of translations which leaves H invariant is equal to £ ⊕ £i.

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